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Neutrino Wave Equation in the Robertson–Walker Geometry

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The massless Dirac equation is separated in the Robertson–Walker geometry. The Schrödinger-like one-dimensional equation to which the problem is reduced is shown to admit a discrete positive spectrum. The existence or nonexistence of the discrete neutrino energy spectrum is connected, in the case of the standard cosmology, with the assumption that the universe is closed or not.

1. INTRODUCTION

The formulation of the quantum wave equation in general relativity is based on the spinorial formalism developed in curved space-time in the pioneering paper by Newman and Penrose (1962). An account of this formalism can be found in the book by Chandrasekhar (1983).

The neutrino wave equation in curved space-time, considered as the massless case of the Dirac equation, was separated in Kerr geometry directly (Unruh, 1973; Teukolsky, 1973) or as a consequence of the separation of the Dirac equation (Chandrasekhar, 1976, 1983). The study of neutrino waves has been considered in the context of Schwarschild's geometry (Brill and Wheeler, 1957).

In this paper we perform the separation of the massless Dirac equation in the Robertson–Walker geometry. This is of interest because the Robertson–Walker metric is generally assumed as a basis of the standard cosmological model (see, for instance, Weinberg, 1972; Kolb and Turner 1990).

The separation of the equation is done along the lines of Teukolsky (1973) and Chandrasekhar (1983). The angular part is explicitly integrated

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and the eigenvalues are determined by generalizing the usual method of integration of the angular part of the Schrödinger equation.

The problem is then reduced to the study of a Schrödinger-like one-dimensional eigenvalue problem, to which qualitative standard mathematical methods apply.

The result is the existence of a discrete positive neutrino energy spectrum in the case of the Robertson–Walker metric which corresponds to the closed universe, provided that the standard cosmology is assumed. In the case of the Robertson–Walker metric which corresponds to the open universe there exists discrete neutrino energy levels, but they are negative, while in the case corresponding to the flat universe the spectrum is continuous and coincides with the positive real line.

The discrete positive spectrum is then determined in a case of physical interest. The separation of the energy levels and their values are, however, found to lie beyond the present experimental sensitivity.

2. NEUTRINO EQUATION IN ROBERTSON–WALKER METRIC

The formulation of the Dirac equation in the Robertson–Walker metric, given by

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-ar^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{1}$$

is given here in the context of the formalism of Newman and Penrose (1962). [For notations, sign conventions, and development of this formalism, see Chandrasekhar (1983).] This requires we define a null-tetrad frame, which we choose to be

$$\begin{aligned}
e^i_{(1)} &\equiv l^i = \frac{1}{\sqrt{2}}\left(1, \frac{(1-ar^2)^{1/2}}{R}, 0, 0\right) \\
e^i_{(2)} &\equiv n^i = \frac{1}{\sqrt{2}}\left(1, -\frac{(1-ar^2)^{1/2}}{R}, 0, 0\right) \\
e^i_{(3)} &\equiv m^i = \frac{1}{\sqrt{2}rR}(0, 0, 1, i\csc\theta) \\
e^i_{(4)} &\equiv m^{*i} = (m^i)^* = \frac{1}{\sqrt{2}rR}(0, 0, 1, -i\csc\theta)
\end{aligned} \tag{2}$$

and calculate the corresponding spin rotation coefficients defined by

$$\gamma_{(a)(b)(c)} = \frac{1}{2}\left[\lambda_{(a)(b)(c)} + \lambda_{(c)(a)(b)} - \lambda_{(b)(c)(a)}\right]$$

where

$$\lambda_{(a)(b)(c)} = e_{(b)i,j}[e^{i}_{(a)}e^{j}_{(b)} - e^{j}_{(a)}e^{i}_{(b)}]$$

In the present case we have

$$\begin{aligned}
\kappa &= \gamma_{311} = 0, \qquad \nu = \gamma_{242} = 0 \\
\lambda &= \gamma_{244} = 0, \qquad \tau = \gamma_{312} = 0 \\
\pi &= \gamma_{241} = 0, \qquad \sigma = \gamma_{313} = 0 \\
\rho &= \gamma_{314} = -\frac{1}{\sqrt{2}}\left[\frac{\dot{R}}{R} + \frac{(1-ar^2)^{1/2}}{rR}\right] \\
\mu &= \gamma_{243} = \frac{1}{\sqrt{2}}\left[\frac{\dot{R}}{R} - \frac{(1-ar^2)^{1/2}}{rR}\right] \\
\beta &= -\alpha = \frac{1}{2}(\gamma_{213} + \gamma_{343}) = \frac{\cot\theta}{2\sqrt{2}rR} \\
\epsilon &= -\gamma = \frac{1}{2}(\gamma_{211} + \gamma_{341}) = \frac{\dot{R}}{2\sqrt{2}R}
\end{aligned} \tag{3}$$

(the dot denotes time derivative).

As a generalization of the Minkowski space-time case, the Dirac equation can be formulated in terms of covariant derivatives and generalized Pauli matrices:

$$\begin{aligned}
\sigma^{i}_{AB'}P^{A}_{;i} + i\mu_{*}\bar{Q}^{C'}\epsilon_{C'B'} &= 0 \\
\sigma^{i}_{AB'}Q^{A}_{;i} + i\mu_{*}\bar{P}^{C'}\epsilon_{C'B'} &= 0
\end{aligned} \tag{4}$$

where

$$\sigma^{i}_{AB'} = \frac{1}{\sqrt{2}}\begin{pmatrix} l^i & m^i \\ m^{*i} & n^i \end{pmatrix} \tag{5}$$

The wave function is represented by the spinors P^A, $\bar{Q}^{A'}$, with $\mu_{*}\sqrt{2}$ being the mass of the particle (Chandrasekhar, 1983).

The Dirac equation (4) can be expressed explicitly in terms of the spin coefficients defined above and in terms of the direction derivatives $D = l^i\partial_i$, $\Delta = n^i\partial_i$, $\delta = m^i\partial_i$, $\delta^* = m^{*i}\partial_i$. In the case of zero-rest-mass particles one gets (Chandrasekhar, 1983, Chapter 10)

$$\begin{aligned}
(D + \epsilon - \rho)F_1 + (\delta^* - \alpha)F_2 &= 0 \\
(\Delta + \mu - \gamma)F_2 + (\delta + \beta)F_1 &= 0 \\
(D + \epsilon - \rho)G_2 - (\delta - \alpha)G_1 &= 0 \\
(\Delta + \mu - \gamma)G_1 - (\delta^* + \beta)G_2 &= 0
\end{aligned} \tag{6}$$

where $F_1 = P^0$, $F_2 = P^1$, $G_1 = \bar{Q}^{1'}$, $G_2 = -\bar{Q}^{0'}$, and the explicit values of the directional derivatives and of the spin coefficients (3) are assumed to hold. Equations (6) are, in the Newman–Penrose formalism, the massless neutrino equations in the Robertson–Walker geometry.

3. SEPARATION OF THE EQUATIONS

Owing to the special dependence of the directional derivatives and of the spin rotation coefficients on the variable ϕ, the ϕ dependence of the wave function is given by $e^{im\phi}$, $m = 0, \pm 1, \pm 2, \pm 3, \ldots$. With this choice, equations (6) become

$$\begin{aligned} &D(rRF_1) + \epsilon rRF_1 + L^- F_2 = 0 \\ &\Delta(rRF_2) + \epsilon rRF_2 + L^+ F_1 = 0 \\ &D(rRG_2) + \epsilon rRG_2 - L^+ G_1 = 0 \\ &\Delta(rRG_1) + \epsilon rRG_1 - L^- G_2 = 0 \end{aligned} \tag{7}$$

where

$$L^{\pm} = \frac{1}{\sqrt{2}}\left(\partial_\theta \mp m \csc\theta + \frac{1}{2}\cos\theta\right) \tag{8}$$

The wave function depends now on the variables r, θ, t. By setting

$$\begin{aligned} F_1 &= A_1(r,\theta)T(t), \qquad G_1 = B_1(r,\theta)T(t) \\ F_2 &= A_2(r,\theta)T(t), \qquad G_2 = B_2(r,\theta)T(t) \end{aligned} \tag{9}$$

and by using the explicit value of ϵ and of the directional derivatives, we can manipulate equations (7) to obtain

$$\begin{aligned} ik &= -\frac{1}{T}(\dot{T}R) - \frac{\dot{R}}{2} \\ ik &= \frac{(rA_1)'(1-ar^2)^{1/2} + \sqrt{2}L^- A_2}{rA_1} \\ ik &= \frac{-(rA_2)'(1-ar^2)^{1/2} + \sqrt{2}L^+ A_1}{rA_2} \\ ik &= \frac{(rB_2)'(1-ar^2)^{1/2} - \sqrt{2}L^+ B_1}{rB_2} \\ ik &= \frac{-(rB_1)'(1-ar^2)^{1/2} - \sqrt{2}L^+ B_2}{rB_1} \end{aligned} \tag{10}$$

where k is a constant of separation, and the prime denotes the ordinary derivative with respect to the variable r.

The time dependence is then

$$T(t) = \frac{C_0}{R(t)^{3/2}} \exp\left[-ik \int_0^t \frac{dt}{R(t)} \right] \tag{11}$$

C_0 is a constant of integration. The explicit time dependence of $R(t)$ and hence of $T(t)$ is connected with the particular cosmological model one assumes. However, as will be seen, all the other qualitative results are unaffected if one chooses the Friedmann model or any other cosmological dynamics governing $R(t)$.

The separation of the equation can be further carried out by setting in (10)

$$\begin{aligned} rA_1 &= S_1(\theta) f_1(r) \\ rA_2 &= S_2(\theta) f_2(r) \\ rB_1 &= S_1(\theta) f_2(r) \\ rB_2 &= S_2(\theta) f_1(r) \end{aligned} \tag{12}$$

One then gets the following equations:

$$\begin{aligned} \sqrt{2} L^- S_2 &= -\lambda S_1 \\ \sqrt{2} L^+ S_1 &= \lambda S_2 \end{aligned} \tag{13}$$

and

$$\begin{aligned} r f_1' (1 - ar^2)^{1/2} - ikr f_1 &= \lambda f_1 \\ r f_2' (1 - ar^2)^{1/2} + ikr f_2 &= \lambda f_2 \end{aligned} \tag{14}$$

λ is a separation constant. As a consequence of (13), S_1 and S_2 satisfy the eigenvalue equations

$$\begin{aligned} 2L^- L^+ S_1 &= -\lambda^2 S_1 \\ 2L^+ L^- S_2 &= -\lambda^2 S_2 \end{aligned} \tag{15}$$

For the radial equation, if we define

$$r_* = \int_0^r \frac{dr}{(1 - ar^2)^{1/2}} \tag{16}$$

equations (14) become:

$$\begin{aligned} \left(\frac{d}{dr_*} - ik \right) f_1 &= \frac{\lambda}{r} f_2 \\ \left(\frac{d}{dr_*} + ik \right) f_2 &= \frac{\lambda}{r} f_1 \end{aligned} \tag{17}$$

or also

$$\left(\frac{d^2}{dr_*^2}+k^2\right)Z_\pm = V_\pm Z_\pm$$
$$V_\pm = \frac{\lambda^2}{r^2} \mp \frac{\lambda}{r^2}\frac{dr}{dr_*}, \qquad Z_\pm = f_1 \pm f_2 \tag{18}$$

If $a=1$, then $r_* = \sin^{-1} r$, so that

$$V_\pm = \frac{\lambda}{\sin^2 r_*}(\lambda \mp \cos r_*) \qquad \left(0 \le r_* \le \frac{\pi}{2}\right) \tag{19}$$

If $a=0$, then $r_* = r$, so that

$$V_\pm = \frac{\lambda}{r^2}(\lambda \mp 1) \qquad (r \ge 0) \tag{20}$$

If $a=-1$, then $r_* = \sinh^{-1} r$, and

$$V_\pm = \frac{\lambda}{\sinh^2 r_*}(\lambda \mp \cosh r_*) \qquad (r_* \ge 0) \tag{21}$$

In every case, $Z_\pm$ satisfies a one-dimensional quantumlike equation.

4. THE ANGULAR EQUATION

We now determine the possible values of λ^2. By using the definition (8) of $L^\pm$, we find that the second equation (15) becomes

$$S'' + S'\cot\theta + S\left[\frac{-4m\cos\theta - 2 - 4m^2 + \cos^2\theta}{4\sin^2\theta}\right] + \lambda^2 S = 0 \tag{22}$$

The first equation (15) gives equation (22) with $m \to -m$.

We are looking for solutions $S(\theta)$ of equation (22) which are regular in $\theta = 0$ and $\theta = \pi$. Let us first consider the case $m \ge 1$. By putting

$$S(\theta) = (1-\xi)^{m/2+1/4}(1+\xi)^{m/2-1/4}f(\xi), \qquad \xi = \cos\theta \tag{23}$$

one readily gets for $f(\xi)$ the equation

$$(1-\xi^2)f'' - [2(m+1)\xi + 1]f' + \left[\lambda^2 - \left(m+\frac{1}{2}\right)^2\right]f = 0 \tag{24}$$

whose acceptable solution, corresponding to $\lambda^2 = (l+1/2)^2$, can be written in terms of Jacobi polynomials (Erdelyi *et al.*, 1953)

$$f(\xi) = P_{l-m}^{(m+1/2,m-1/2)}(\xi), \qquad m \ge 1, \quad l = m, m+1, m+2, \ldots \tag{25}$$

Therefore

$$S_{l,m}(\theta) = (1-\cos\theta)^{m/2+1/4}(1+\cos\theta)^{m/2-1/4}P_{l-m}^{(m+1/2,m-1/2)}(\cos\theta) \tag{26}$$

If $m \leq -1$, it suffices to replace [see equation (22)] m by $|m|$ and ξ by $-\xi$. Thus, apart from an irrelevant factor [recall that $P_n^{(\alpha,\beta)}(-x) = (-)^n P_n^{(\beta,\alpha)}(x)$]

$$S_{l,m}(\theta) = (1+\cos\theta)^{|m|/2+1/4}(1-\cos\theta)^{|m|/2-1/4} P_{l-|m|}^{(|m|-1/2,|m|+1/2)}(\cos\theta)$$
$$m \leq -1, \qquad l = |m|, |m|+1, |m|+2, \ldots \tag{27}$$

Finally, if $m = 0$, we put

$$S(\theta) = (1-\xi^2)^{1/4} g(\xi) \tag{28}$$

to obtain, corresponding to $\lambda^2 = (l+1)^2$

$$S_{l,0}(\theta) = (\sin\theta)^{1/2} U_l(\xi), \qquad l = 0, 1, 2, \ldots \tag{29}$$

where $U_l(\xi)$ is the lth Tchebychef polynomial of the second kind (Erdelyi *et al.*, 1953).

5. THE RADIAL EQUATION

The main consequence of the results of the previous sections is the existence of a positive discrete spectrum of k^2. This result follows directly from the analytical behavior of the potentials $V_\pm$ by applying qualitative standard results in the theory of the one-dimensional Schrödinger equation (Reed and Simon, 1978) in confined and nonconfined physical regions. According to these qualitative considerations one can check that discrete values of k^2 do indeed exist in the case $a = -1$ (open universe of the standard cosmology) but that they take negative values. In the case corresponding to the flat universe, $a = 0$, k^2 admit only continuous positive values that coincide with the positive real line. It is a matter of fact that only in the physical case $a = 1$ (corresponding to the closed universe of the Friedmann model) do discrete values of k^2 exist and are positive.

To give explicit results, we study here the case $a = 1$ for the lowest value $l = 0$ of l. By taking into account equations (18), (19), we have to solve the equations

$$\left(-\frac{d^2}{dr_*^2} + \frac{1 \mp \cos r_*}{\sin^2 r_*}\right) Z_\pm = k^2 Z_\pm \tag{30}$$

with the conditions

$$Z_\pm(0) = Z_\pm\left(\frac{\pi}{2}\right) = 0 \tag{31}$$

[which follow from (12), (18) and from (14) recalling that λ and k are independent constants] and $Z_\pm$ in $L^2(0, \pi/2)$. We first study the case relative to Z_+.

By putting

$$Z_+ = (1-\xi)^{1/2}(1+\xi)f, \qquad \xi = \cos r_* \tag{32}$$

equation (30) becomes

$$(1-\xi^2)f'' + (1-4\xi)f' + \left(k^2 - \frac{9}{4}\right)f = 0 \tag{33}$$

with the solution

$$\begin{aligned} Z_+(\xi) = (1+\xi)\Bigg[& A\, {}_2F_1\left(1-k, 1+k; \frac{5}{2}; \frac{1+\xi}{2}\right) \\ & + B\left(\frac{1+\xi}{2}\right)^{-3/2} {}_2F_1\left(-\frac{1}{2}-k, -\frac{1}{2}+k; -\frac{1}{2}; \frac{1+\xi}{2}\right)\Bigg] \end{aligned} \tag{34}$$

The condition $Z_+(1) = 0$ implies

$$\frac{A}{B} = -\frac{2}{3}k(1-4k^2)\tan(\pi k) \tag{35}$$

Furthermore (Erdelyi *et al.*, 1953)

$$\begin{aligned} & {}_2F_1\left(1-k, 1+k; \frac{5}{2}; \frac{1}{2}\right) \\ & = (1-x)^{1/2}\, {}_2F_1\left(\frac{3}{2}-k, \frac{3}{2}+k; \frac{5}{2}; x\right)\Bigg|_{x=1/2} \\ & = (1-x)^{1/2} \frac{6}{1-4k^2} \frac{d}{dx}\, {}_2F_1\left(\frac{1}{2}-k, \frac{1}{2}+k; \frac{3}{2}; x\right)\Bigg|_{x=1/2} \\ & = (1-x)^{1/2} \frac{6}{1-4k^2} \frac{d}{dx} \frac{\sin(2k \sin^{-1}\sqrt{x})}{2k\sqrt{x}}\Bigg|_{x=1/2} \\ & = \frac{3}{1-4k^2}\left[2\cos\left(k\frac{\pi}{2}\right) - \frac{1}{k}\sin\left(k\frac{\pi}{2}\right)\right] \end{aligned} \tag{36}$$

and

$$\begin{aligned} & \frac{d}{dx}\, {}_2F_1\left(-\frac{1}{2}-k, -\frac{1}{2}+k; -\frac{1}{2}; x\right) \\ & = -2\left(\frac{1}{4}-k^2\right) {}_2F_1\left(\frac{1}{2}-k, \frac{1}{2}+k; \frac{1}{2}; x\right) \\ & = -\frac{1}{2}(1-4k^2)\frac{\cos(2k \sin^{-1}\sqrt{x}}{(1-x)^{1/2}} \end{aligned} \tag{37}$$

whence

$$
{}_2F_1\left(-\frac{1}{2}-k,\ -\frac{1}{2}+k;\ -\frac{1}{2};\frac{1}{2}\right)
$$
$$
=\frac{1}{2}(1-4k^2)\int_{1/2}^{1} dx\,(1-x)^{-1/2}\cos(2k\sin^{-1}\sqrt{x})+2k\sin\pi k
$$
$$
=\frac{1}{\sqrt{2}}\left[\cos\left(k\frac{\pi}{2}\right)+2k\sin\left(k\frac{\pi}{2}\right)\right] \tag{38}
$$

Therefore, from (34)–(36) and (38), the condition $Z_+(0)=0$ implies

$$
2k\tan\left(\frac{\pi k}{2}\right)=1 \tag{39}
$$

It follows that there exists a countable set of solutions k_n of (39) whose behavior for large n is

$$
k_n \cong 4n \tag{40}
$$

The equation for Z_-, which follows from the one for Z_+ with $\xi\to-\xi$, gives, as is easily checked, the same spectrum of values of k_n^2.

6. CONCLUDING REMARKS

The separation of the neutrino wave equation in the Robertson–Walker metric carried out in the previous sections reduces the problem to the solution of a Schrödinger-like equation. The radial equation contains a separation constant k^2 which can consistently assume discrete positive values only in the physical case $a=1$, corresponding to the closed universe of the standard cosmology. They are given, for large values, by $k_n^2\cong(4n)^2$.

If the constant k^2 in equation (18) is interpreted as the neutrino energy E_n, then the result is the existence of a discrete neutrino energy spectrum. Since we have performed our calculations in Planck units, a simple numerical estimate gives:

If $E_n=25$ eV, then $E_{n+1}-E_n\cong 10^{-13}$ eV.
If $E_n=4$ MeV, then $E_{n+1}-E_n\cong 10^{-10}$ eV.
If $E_n=20$ GeV, then $E_{n+1}-E_n\cong 10^{-8}$ eV.
If $E_{n+1}-E_n=1$ eV, then $E_n\cong 10^{26}$ eV.
If $E_{n+1}-E_n=10^{-6}$ eV, then $E_n\cong 10^{14}$ eV.

It follows that, when the energy is small, the separation of the energy levels is very tiny to be experimentally tested; when such separation is sensible, the energy involved is very improbable.

In spite of these negative aspects, discrete neutrino energies have a cosmological interest: they could also be a component of the 2 K black-body neutrino distribution which followed the neutrino decoupling in the early universe of the standard cosmology.

If neutrinos would be produced in other different successive ways, then the discrete energy levels could have been reached after some thermalization process.

Moreover, if there were experimental evidence of the nonexistence of such discrete neutrino energies, then the previous considerations would seem to support the idea of a flat universe in the case of the standard cosmology.

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